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Ali Al-Aati
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen
alati2005@yahoo.com

Wedad M. Al-Sharai
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen
alshareiw@gmail.com

Wafa M. Al-Sharai
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen
wafaa.alshrai@gmail.com

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Ali Al-Aati
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen

Wedad M. Al-Sharai
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen

Wafa M. Al-Sharai
Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen

Abstract

In this work, we introduce a new method for solving some partial differential equations called double Aboodh-Kamal transform, some useful properties for the transform are presented. Furthermore, we use this transform for solving some linear partial differential equations and find some functions.

Keywords: Double Aboodh-Kamal Transform, Aboodh transform, Kamal transform, partial differential equations.

1- INTRODUCTION

In previous years, much notice has been given to dealing with the single, double and triple transform [5, 6, 9, 10, 11, 12], which have many applications in various fields of mathematical sciences and engineering such as acoustics, physics, chemistry, etc.,.

In recent years, great attention has been given to deal with the double integral transforms; see for example, [3, 8, 13]. Alfaqeih and Misirli in [4] dealt with double Shehu transform to get the solution of initial and boundary value problems in different areas of real life science and engineering.

Analogous to [3], we applied a new double Aboodh-Kamal transform to solve some partial differential equations subject to the initial and boundary conditions, through the derivation of general formula for solutions of these equations, or by applying the double Aboodh-Kamal transform directly to the given equation. The aim of the present study is to introduce a new method for solving some partial differential equations subject to the initial and boundary conditions called double Aboodh-Kamal transform, the definition of double Aboodh-Kamal transform and its inverse. We also discussed some theorems and properties about the double Aboodh-Kamal transform and gave the double Aboodh-Kamal transform of some elementary functions. Moreover, we implement the double Aboodh-Kamal transform method to some equations.

2- PRELIMINARIES

Definition 2.1. [1] The single Aboodh transform of the real function u(x) of exponential order is de ned over the set of functions

$$\mathcal{M} = \left\{ u(x) : \exists K, \tau_1, \tau_2 > 0, |u(x)| < K e^{|x|\tau_i},\ x \in (-1)^i \times [0,\infty),\ i = 1,2 \right\},$$

by the following integral

$$\begin{aligned} A[u(x)] = F(q) &= \frac{1}{q}\int_0^\infty e^{-qx} u(x)dx \\ &= \frac{1}{q}\lim_{a\to\infty}\int_0^a e^{-qx} u(x)dx, \quad \tau_1 \le q \le \tau_2. \end{aligned}$$

It converges if the limit of the integral exists, and diverges if not. And the inverse Aboodh transform is

$$A^{-1}[F(q)] = u(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} qe^{qx}F(q)dq, \quad \alpha \geq 0.$$

Definition2.2. [2] Let A be functions set defined by

$$\mathcal{A} = \left\{u(t) : \exists M, \gamma_1, \gamma_2 > 0, |u(t)| < Me^{\frac{|t|}{\gamma_j}},\ t \in (-1)^j \times [0,\infty),\ j = 1,2\right\},$$

Where M is a constant and γ_1, γ_2 are finite constants or infinite.

For a function of exponential order, the single Kamal integral transform of the real continuous function $u(t)$ is defined as follows

$$\begin{aligned} K[u(t)] = F(r) &= \int_0^\infty e^{-\frac{t}{r}}u(t)dt \\ &= \lim_{b\to\infty}\int_0^b e^{-\frac{t}{r}}u(t)dt, \end{aligned}$$

Where $e^{-\frac{t}{r}}$ is the kernel function, and K[.] is the Kamal transform operator. Provided that the integral exists.

Moreover, the inverse Kamal transform is defined by

$$u(t) = K^{-1}[F(r)] = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{t}{r}}F(\frac{1}{r})dr, \ \beta \geq 0.$$

In the next definition, we introduce the double Aboodh-Kamal transform.

Definition 2.3. The double Aboodh-Kamal transform of the function $u(x,t)$ of two variables $x > 0$ and $t > 0$ is denoted by $A_xK_t[u(x,t)] = F(q,r)$ and defined as

$$\begin{aligned} A_xK_t[u(x,t)] = F(q,r) &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}u(x,t)dxdt \\ &= \frac{1}{q}\lim_{a\to\infty, b\to\infty}\int_0^a\int_0^b e^{-(qx+\frac{t}{r})}u(x,t)dxdt. \qquad (2.1) \end{aligned}$$

It converges if the limit of the integral exists, and diverges if not.

Definition2.4. The inverse double Aboodh-Kamal transform of a function F(q,r) is given by $A_x^{-1}K_t^{-1}[F(q,r)] = u(x,t)$.

Equivalently,

$$u(x,t) = A_x^{-1}K_t^{-1}[F(q,r)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} qe^{qx}\left(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{t}{r}}F(q,\frac{1}{r})dr\right)dq, \quad (2.2)$$

where α and β are real constants.

3- EXISTENCE AND UNIQUENESS OF THE DOUBLE ABOODH-KAMAL TRANSFORM

Definition3.1. [7] A function $u(x,t)$ is said to be of exponential orders $\lambda, \eta > 0$ on $0 \le x, t < \infty$, if there exists positive constants L, X and T such that

$$|u(x,t)| \le Le^{\lambda x+\eta t},\ \forall\ x > X,\ t > T,$$

and we write

$$u(x,t) = o(e^{\lambda x+\eta t}),\ as\ x \to \infty,\ t \to \infty.$$

Or equivalently,

$$\lim_{x\to\infty, t\to\infty} e^{-(qx+\frac{t}{r})}|u(x,t)| \le L \lim_{x\to\infty, t\to\infty} e^{-(q-\lambda)x}e^{-(\frac{1}{r}-\eta)t} = 0,\ q > \lambda,\ \frac{1}{r} > \eta.$$

Theorem3.2. [5] Let $u(x,t)$ be a continuous function in every finite intervals (0,X) and (0,T) and of exponential order $e^{(\lambda x+\eta t)}$ then the double Aboodh-Kamal transform of $u(x,t)$ exists for $all\ q > \lambda\ and\ \frac{1}{r} > \eta$.

Proof. Let $u(x,t)$ be of exponential order $e^{(\lambda x+\eta t)}$ such that

$$\left|u(x,t)\right| \le Le^{(\lambda x+\eta t)},\ \forall\ x > X,\ t > T.$$

Then, we have

$$\begin{aligned} \left|F(q,r)\right| &= \left|\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}u(x,t)dxdt\right| \\ &\le \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}|u(x,t)|dxdt \\ &\le \frac{L}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{(\lambda x+\eta t)}dxdt \\ &= \frac{L}{q}\int_0^\infty e^{-(q-\lambda)x}dx\int_0^\infty e^{-(\frac{1}{r}-\eta)t}dt \\ &= \frac{Lr}{q(q-\lambda)(1-\eta r)}. \end{aligned}$$

Thus, the proof is complete.

Theorem3.3. Let $F_1(q,r)$ and $F_2(q,r)$ be the double Aboodh-Kamal transform of the continuous functions $u_1(x,t)$ and $u_2(x,t)$ defined for $x,t \ge 0$ *respectively.*

If $F_1(q,r) = F_2(q,r)$, *then* $u_1(x,t) = u_2(x,t)$.

Proof. Assume that α and β are sufficiently large, since

$$u(x,t) = A_x^{-1}K_t^{-1}[F(q,r)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} qe^{qx}\left(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{t}{r}}F(q,\frac{1}{r})dr\right)dq,$$

we deduce that

$$\begin{aligned} u_1(x,t) &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} qe^{qx}\left(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{t}{r}}F_1(q,\frac{1}{r})dr\right)dq \\ &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} qe^{qx}\left(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{t}{r}}F_2(q,\frac{1}{r})dr\right)dq \\ &= u_2(x,t). \end{aligned}$$

This ends the proof of the theorem.

4- SOME USEFUL PROPERTIES OF THE DOUBLE ABOODH-KAMAL TRANSFORM

4.1. Linearity property. If the double Aboodh-Kamal transform of functions $u_1(x,t)$ and $u_2(x,t)$ are $F_1(q,r)$ and $F_2(q,r)$ respectively, then the double Aboodh-Kamal transform of $\alpha u_1(x,t) + \beta u_2(x,t)$ is given by $\alpha F_1(q,r) + \beta F_2(q,r)$, where α and β are arbitrary constants.

Proof.

$$\begin{aligned} A_xK_t[\alpha u_1(x,t) + \beta u_2(x,t)] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\Big(\alpha u_1(x,t) + \beta u_2(x,t)\Big)dxdt \\ &= \frac{\alpha}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}u_1(x,t)dxdt \\ &+ \frac{\beta}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}u_2(x,t)dxdt \\ &= \alpha F_1(q,r) + \beta F_2(q,r)\,. \end{aligned} \tag{4.1}$$

4.2. Shifting property. If the double Aboodh-Kamal transform of function $u(x,t)$ is $F(q,r)$, then for any pair of real constants α, $\beta > 0$,

$$A_xK_t\left[e^{(\alpha x+\beta t)}u(x,t)\right] = \frac{(q-\alpha)}{q}F\left(q-\alpha, \frac{r}{1-\beta r}\right) \tag{4.2}$$

Proof.

$$\begin{aligned} A_xK_t[e^{(\alpha x+\beta t)}u(x,t)] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{(\alpha x+\beta t)}e^{-(qx+\frac{t}{r})}u(x,t)dxdt \\ &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-\left((q-\alpha)x+(\frac{1-\beta r}{r})t\right)}u(x,t)dxdt \\ &= \frac{(q-\alpha)}{q}F\left(q-\alpha,\frac{r}{1-\beta r}\right). \end{aligned} \quad (4.3)$$

4.3. Change of scale property. If the double Aboodh-Kamal transform of the function $u(x,t)$ is $F(q,r)$, then the double Aboodh-Kamal transform of function $u(\alpha x,\beta t)$ is given by $\frac{1}{\alpha^2\beta}F(\frac{q}{\alpha},\beta r)$.

Proof. Using the definition of double Aboodh-Kamal transform, we get

$$A_xK_t[u(\alpha x,\beta t)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}u(\alpha x,\beta t)dxdt.$$

Let $v=\alpha x,\ \tau=\beta t$, then

$$\begin{aligned} A_xK_t[u(\alpha x,\beta t)] &= \frac{1}{\alpha\beta q}\int_0^\infty\int_0^\infty e^{-\frac{qv}{\alpha}}e^{-\frac{\tau}{\beta r}}u(v,\tau)dvd\tau \\ &= \frac{1}{\alpha^2\beta}F(\frac{q}{\alpha},\beta r). \end{aligned} \quad (4.4)$$

4.4. Derivatives properties.

□

4.4. **Derivatives properties.** If $A_xK_t[u(x,t)] = F(q,r)$, then

$$(1)\ A_xK_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] = qF(q,r)-\frac{1}{q}K[u(0,t)]. \quad (4.5)$$

Proof.

$$\begin{aligned} A_xK_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial u(x,t)}{\partial x}dxdt \\ &= \int_0^\infty e^{-\frac{t}{r}}dt\Big(\frac{1}{q}\int_0^\infty e^{-qx}u_x(x,t)dx\Big). \end{aligned}$$

Using integration by parts, let $\mathsf{u}=e^{-qx},\ \mathsf{dv}=u_x(x,t)dx$, then we obtain

$$\begin{aligned} A_xK_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] &= \int_0^\infty e^{-\frac{t}{r}}dt\Big\{\frac{1}{q}\Big(-u(0,t)+q\int_0^\infty e^{-qx}u(x,t)dx\Big)\Big\} \\ &= qF(q,r)-\frac{1}{q}K[u(0,t)]. \end{aligned}$$

□

$$(2)\ A_xK_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] = \frac{1}{r}F(q,r)-A[u(x,0)]. \quad (4.6)$$

Proof.

$$A_xK_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial u(x,t)}{\partial t}dxdt = \frac{1}{q}\int_0^\infty e^{-qx}dx\Big(\int_0^\infty e^{-\frac{t}{r}}u_t(x,t)dt\Big).$$

Using integration by parts, let $\mathrm{u} = e^{-\frac{t}{r}}$, $d\mathrm{v} = u_t(x,t)dt$, then we obtain

$$A_xK_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] = \frac{1}{q}\int_0^\infty e^{-qx}dx\Big(-u(x,0) + \frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}u(x,t)dt\Big) = \frac{1}{r}F(q,r) - A[u(x,0)].$$

□

$$(3)\ A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] = q^2F(q,r) - K[u(0,t)] - \frac{1}{q}K[u_x(0,t)]. \quad (4.7)$$

Proof.

$$A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial^2 u(x,t)}{\partial x^2}dxdt = \int_0^\infty e^{-\frac{t}{r}}dt\Big(\frac{1}{q}\int_0^\infty e^{-qx}u_{xx}(x,t)dx\Big).$$

Integration by parts twice, we obtain

$$A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] = \int_0^\infty e^{-\frac{t}{r}}dt\Big(\frac{1}{q}\Big\{-u_x(0,t) + q\big\{-u(0,t) + q^2\int_0^\infty e^{-qx}u(x,t)dx\big\}\Big\}\Big) = q^2F(q,r) - K[u(0,t)] - \frac{1}{q}K[u_x(0,t)].$$

□

$$(4)\ A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big] = \frac{1}{r^2}F(q,r) - \frac{1}{r}A[u(x,0)] - A[u_t(x,0)]. \quad (4.8)$$

Proof.

$$A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial^2 u(x,t)}{\partial t^2}dxdt = \frac{1}{q}\int_0^\infty e^{-qx}dx\Big(\int_0^\infty e^{-\frac{t}{r}}u_{tt}(x,t)dt\Big).$$

Integration by parts twice, we obtain

$$A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big] = \frac{1}{q}\int_0^\infty e^{-qx}dx\Big(-u_t(x,0) - \frac{1}{r}u(x,0) + \frac{1}{r^2}\int_0^\infty e^{-\frac{t}{r}}u(x,t)dt\Big) = \frac{1}{r^2}F(q,r) - \frac{1}{r}A[u(x,0)] - A[u_t(x,0)].$$

□

$$(5)\ A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big] = \frac{q}{r}F(q,r) - qA[u(x,0)] - \frac{1}{q}K[u_t(0,t)]. \quad (4.9)$$

Proof.

$$\begin{aligned} A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial^2 u(x,t)}{\partial x\partial t}dxdt \\ &= \int_0^\infty e^{-\frac{t}{r}}dt\Big(\frac{1}{q}\int_0^\infty e^{-qx}u_{xt}(x,t)dx\Big). \end{aligned}$$

Integration by parts twice, we obtain

$$\begin{aligned} A_xK_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big] &= \int_0^\infty e^{-\frac{t}{r}}dt\Big(\frac{1}{q}\Big\{-u_t(0,t)+q\int_0^\infty e^{-qx}u_t(x,t)dx\Big\}\Big) \\ &= -\frac{1}{q}K[u_t(0,t)]+\frac{q}{q}\int_0^\infty e^{-qx}dx\int_0^\infty e^{-\frac{t}{r}}u_t(x,t)dt \\ &= -\frac{1}{q}K[u_t(0,t)]+\frac{q}{q}\int_0^\infty e^{-qx}dx\Big(-u(x,0)+\frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}u(x,t)dt\Big) \\ &= \frac{q}{r}F(q,r)-qA[u(x,0)]-\frac{1}{q}K[u_t(0,t)]. \end{aligned}$$

5. THE DOUBLE ABOODH-KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS

(1) If the function $u(x,t)=1$, then

$$A_xK_t[u(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}dxdt=\frac{r}{q^2}. \tag{5.1}$$

(2) If the function $u(x,t)=xt$, then

$$A_xK_t[u(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}xtdxdt=\frac{r^2}{q^3}. \tag{5.2}$$

(3) If the function $u(x,t)=x^nt^m,\ n,m=0,1,2,\ \ldots$, then

$$A_xK_t[u(x,t)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}x^nt^mdxdt=n!m!\frac{r^{m+1}}{q^{n+2}}. \tag{5.3}$$

(4) If the function $u(x,t)=x^\sigma t^\nu,\ \sigma\geq-1,\ \nu\geq-1$, then

$$A_xK_t[u(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}x^\sigma t^\nu dxdt=\frac{1}{q}\int_0^\infty e^{-qx}x^\sigma dx\int_0^\infty e^{-\frac{t}{r}}t^\nu dt.$$

Let $\xi=qx$ and $\eta=\frac{t}{r}$, then we have

$$\begin{aligned} A_xK_t[u(x,t)] &= \frac{1}{q^{\sigma+2}}\int_0^\infty e^{-\xi}\xi^\sigma d\xi\Big(r^{\nu+1}\int_0^\infty e^{-\eta}\eta^\nu d\eta\Big) \\ &= \frac{\Gamma(\sigma+1)}{q^{\sigma+2}}\Gamma(\nu+1)r^{\nu+1}. \end{aligned} \tag{5.4}$$

Where $\Gamma(.)$ is the Euler gamma function.

(5) If the function $u(x,t)=e^{\alpha x+\beta t}$, then

$$A_xK_t[u(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{\alpha x+\beta t}dxdt=\frac{r}{q(q-\alpha)(1-\beta r)}. \tag{5.5}$$

(6) If the function $u(x,t)=\sin(\alpha x+\beta t)$, then

$$\begin{aligned} A_xK_t[u(x,t))] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\sin(\alpha x+\beta t)dxdt \\ &= \frac{r(\alpha+\beta qr)}{q(q^2+\alpha^2)(1+\beta^2r^2)}. \end{aligned} \tag{5.6}$$

(7) If the function $u(x,t)=\cos(\alpha x+\beta t)$, then

$$\begin{aligned} A_xK_t[u(x,t)] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\cos(\alpha x+\beta t)dxdt \\ &= \frac{r(q-\alpha\beta r)}{q(q^2+\alpha^2)(1+\beta^2r^2)}. \end{aligned} \tag{5.7}$$

Consequently,

$$A_xK_t[\sinh(\alpha x+\beta t)] = \frac{r(\alpha+\beta qr)}{q(q^2-\alpha^2)(1-\beta^2r^2)}, \tag{5.8}$$

$$A_xK_t[\cosh(\alpha x+\beta t)] = \frac{r(q+\alpha\beta r)}{q(q^2-\alpha^2)(1-\beta^2r^2)}. \tag{5.9}$$

6- APPLICATIONS

In this section, we apply the double Aboodh-Kamal transform method to solve linear partial differential equations. Let the second-order nonhomogeneous partial differential equation in two independent variables be in the form

$$Bu_{xx}(x,t)+Cu_{tt}(x,t)+Du_x(x,t)+Eu_t(x,t)+Gu(x,t)=h(x,t),(x,t)\in\mathbb{R}_+^2, \tag{6.1}$$

with the initial conditions

$$u(x,0)=\hbar_1(x), \qquad u_t(x,0)=\hbar_2(x), \tag{6.2}$$

and the boundary conditions

$$u(0,t)=\hbar_3(t), \qquad u_x(0,t)=\hbar_4(t), \tag{6.3}$$

where B, C, D, E and G are constants and *h(x, t)* is the source term.

Using the property of partial derivative of the double Aboodh-Kamal transform for equation (6.1), single Aboodh transform for equation (6.2) and single Kamal transform for equation (6.3) and simplifying, we obtain that

$$F(q,r)=\left[\frac{(\frac{C}{r}+E)\hbar_1(q)+C\hbar_2(q)+(B+\frac{D}{q})\hbar_3(r)+\frac{B}{q}\hbar_4(r)+H(q,r)}{(Bq^2+\frac{C}{r^2}+Dq+\frac{E}{r}+G)}\right], \tag{6.4}$$

where $H(q,r)=A_xk_t[h(x,t)]$.

Finally, solving this algebraic equation in F (q, r) and taking the inverse double Aboodh-Kamal transform on both sides of equation (6.4), yields

$$u(x,t)=A_x^{-1}K_t^{-1}\left[\frac{(\frac{C}{r}+E)\hbar_1(q)+C\hbar_2(q)+(B+\frac{D}{q})\hbar_3(r)+\frac{B}{q}\hbar_4(r)+H(q,r)}{(Bq^2+\frac{C}{r^2}+Dq+\frac{E}{r}+G)}\right], \tag{6.5}$$

which represent the general formula for the solution of equation (6.1) by the double Aboodh- Kamal transform method.

Example 6.1. *Use double Aboodh-Kamal transform method to solve the following first order nonhomogeneous partial differential equation*

$$u_x(x,t) + u_t(x,t) = 3x^2 + 3t^2, \tag{6.6}$$

subject to the conditions

$$u(x,0) = x^3, \qquad u(0,t) = t^3. \tag{6.7}$$

Solution.
Applying the double Aboodh-Kamal transform on both sides of equation (6.6), we have

$$A_x K_t[u_x(x,t) + u_t(x,t)] = A_x K_t\left[3x^2 + 3t^2\right].$$

By linearity property and partial derivative properties of double Aboodh-Kamal transform, we get

$$qF(q,r) - \frac{1}{q}K[u(0,t)] + \frac{1}{r}F(q,r) - A[u(x,0)] = \frac{6r}{q^4} + \frac{6r^3}{q^2}, \tag{6.8}$$

where

$$A_x K_t\left[3x^2 + 3t^2\right] = \frac{6r}{q^4} + \frac{6r^3}{q^2}.$$

Substituting

$$\hbar_1(q) = \frac{6}{q^5},\ \hbar_3(r) = 6r^4,$$

into equation (6.8) and simplify to obtain

$$\left(\frac{qr+1}{r}\right)F(q,r) = \frac{6(qr+1)}{q^5} + \frac{6r^3(qr+1)}{q^2},$$

or equivalently,

$$F(q,r) = \frac{6(qr+1)}{q^5}\left(\frac{r}{qr+1}\right) + \frac{6r^3(qr+1)}{q^2}\left(\frac{r}{qr+1}\right) = \frac{6r}{q^5} + \frac{6r^4}{q^2}. \tag{6.9}$$

Taking the inverse double Aboodh-Kamal transform of equation (6.9), we get

$$u(x,t) = A_x^{-1}K_t^{-1}\left[\frac{6r}{q^5} + \frac{6r^4}{q^2}\right] = x^3 + t^3.$$

Which is the required solution of (6.6).

Example 6.2. *Consider the following linear problem*

$$u_{xt}(x,t) = -x + u(x,t), \tag{6.10}$$

subject to the conditions

$$u(x,0) = x + e^x, \qquad u_t(0,t) = e^t.$$

Solution.
Applying the double Aboodh-Kamal transform on both sides of equation (6.10), we get

$$A_x K_t[u_{xt}(x,t)] = A_x K_t[-x + u(x,t)].$$

By linearity property and partial derivative properties of double Aboodh-Kamal transform, we get

$$\frac{q}{r}F(q,r) - qA[u(x,0)] - \frac{1}{q}K[u_t(0,t)] = -\frac{r}{q^3} + F(q,r). \tag{6.11}$$

Substituting

$$A[u(x,0)] = \frac{1}{q^3} + \frac{1}{q(q-1)}, \quad K[u_t(0,t)] = \frac{r}{1-r},$$

in (6.11) and simplifying, we get

$$\left(\frac{q-r}{r}\right)F(q,r) = \frac{q-r}{q(q-1)(1-r)} + \frac{q-r}{q^3},$$

or equivalently,

$$F(q,r) = \frac{1}{q(q-1)}\frac{r}{(1-r)} + \frac{r}{q^3}. \tag{6.12}$$

Taking the inverse double Aboodh-Kamal transform of equation (6.12), we get the solution of (6.10)

$$\begin{aligned} u(x,t) &= A_x^{-1}K_t^{-1}\left[\frac{1}{q(q-1)}\frac{r}{(1-r)} + \frac{r}{q^3}\right] \\ &= e^{x+t} + x. \end{aligned}$$

Example 6.3. Consider the following boundary Poisson equation

$$u_{xx}(x,t) + u_{tt}(x,t) = t\sin x, \tag{6.13}$$

subject to the conditions

$$\begin{aligned} u(x,0) &= 0, \qquad u_t(x,0) = -\sin x, \\ u(0,t) &= 0, \qquad u_x(0,t) = -t. \end{aligned}$$

Solution.

Applying the double Aboodh-Kamal transform on both sides of equation (6.13), we get

$$q^2F(q,r)-K[u(0,t)]-\frac{1}{q}K[u_x(0,t)]+\frac{1}{r^2}F(q,r)-\frac{1}{r}A[u(x,0)]-A[u_t(x,0)]=\frac{r^2}{q(q^2+1)}. \quad (6.14)$$

Using given initial conditions and arrangement, equation (6.14) becomes

$$\begin{aligned} F(q,r) &= \frac{r^2}{q^2r^2+1}\left\{\frac{r^2}{q(q^2+1)}-\frac{r^2}{q}-\frac{1}{q(q^2+1)}\right\} \\ &= \frac{-r^2}{q(q^2+1)}. \qquad (6.15) \end{aligned}$$

Taking the inverse double Aboodh-Kamal transform of equation (6.15), we get the solution of (6.13)

$$\begin{aligned} u(x,t) &= A_x^{-1}K_t^{-1}\left[\frac{-r^2}{q(q^2+1)}\right] \\ &= -t\sin x. \end{aligned}$$

Example 6.4. Consider the following partial differential equation

$$u_{xx}(x,t)-u_{tt}(x,t)=0, \quad (6.16)$$

subject to the conditions

$$\begin{aligned} u(x,0)=\sin x=\hbar_1(x), &\qquad u_t(x,0)=2=\hbar_2(x), \\ u(0,t)=2t=\hbar_3(t), &\qquad u_x(0,t)=\cos t=\hbar_4(t). \end{aligned}$$

Solution.

Applying the double Aboodh-Kamal transform on both sides of equation (6.16), we get

$$q^2F(q,r)-K[u(0,t)]-\frac{1}{q}K[u_x(0,t)]-\frac{1}{r^2}F(q,r)+\frac{1}{r}A[u(x,0)]+A[u_t(x,0)]=0. \quad (6.17)$$

Substituting

$$\begin{aligned} A[\hbar_1(q)]=\frac{1}{q(q^2+1)}, &\qquad A[\hbar_2(q)]=\tfrac{2}{q^2}, \\ K[\hbar_3(r)]=2r^2, &\qquad K[\hbar_4(r)]=\tfrac{r}{1+r^2}, \end{aligned}$$

in (6.17) and simplifying, we get

$$F(q,r)=\frac{1}{q(q^2+1)}\frac{r}{(1+r^2)}+\frac{2r^2}{q^2}. \quad (6.18)$$

Taking the inverse double Aboodh-Kamal transform of equation (6.18), we get the solution of (6.16)

$$\begin{aligned} u(x,t) &= A_x^{-1}K_t^{-1}\left[\frac{1}{q(q^2+1)}\frac{r}{(1+r^2)}+\frac{2r^2}{q^2}\right] \\ &= \sin x\cos t+2t. \end{aligned}$$

Example 6.5. Consider the following linear telegraph equation in the form

$$u_{xx}(x,t)=u_{tt}(x,t)+u_t(x,t)+u(x,t), \qquad (6.19)$$

with the conditions

$$\begin{aligned} u(x,0) &= e^x=\hbar_1(x),\ u_t(x,0) &= -e^x=\hbar_2(x), \\ u(0,t) &= e^{-t}=\hbar_3(t),\ u_x(0,t) &= e^{-t}=\hbar_4(t). \end{aligned}$$

Solution.

Substituting

$$\hbar_1(q)=\frac{1}{q(q-1)},\ \hbar_2(q)=\frac{-1}{q(q-1)},\ \hbar_3(r)=\frac{r}{(1+r)},\ \hbar_4(r)=\frac{r}{(1+r)},$$

in (6.5) and simplifying, we get the solution of (6.19)

$$\begin{aligned} u(x,t) &= A_x^{-1}K_t^{-1}\left[\frac{r^2}{(q^2r^2-r^2-r-1)}\frac{(q^2r^2-r^2-r-1)}{qr(q-1)(1+r)}\right] \\ &= A_x^{-1}K_t^{-1}\left[\frac{1}{q(q-1)}\frac{r}{(1+r)}\right] \\ &= e^{x-t}. \end{aligned}$$

Example 6.6. Use the double Aboodh-Kamal transform method to solve the nonhomogeneous partial differential equation

$$u_t(x,t)-u_{xx}(x,t)=-6x, \qquad (6.20)$$

subject to the conditions

$$\begin{aligned} u(x,0)=x^3+\sin x=\hbar_1(x),\ u_t(x,0) &= -\sin x=\hbar_2(x), \\ u(0,t)=0=\hbar_3(t),\ u_x(0,t) &= e^{-t}=\hbar_4(t). \end{aligned}$$

Solution.

Applying the double Aboodh-Kamal transform on both sides of equation (6.20), we have

$$A_xK_t[u_t(x,t)-u_{xx}(x,t)]=A_xK_t[-6x].$$

By linearity property and partial derivative properties of double Aboodh-

Kamal transform, we get

$$\frac{1}{r}F(q,r) - A[u(x,0)] - q^2F(q,r) + K[u(0,t)] + \frac{1}{q}K[u_x(0,t)] = -\frac{6r}{q^3}. \tag{6.21}$$

Substituting

$$\hbar_1(q) = \frac{6}{q^5} + \frac{1}{q(q^2+1)},\ \hbar_3(r) = 0,\ \hbar_4(r) = \frac{r}{(1+r)},$$

in (6.21) and simplifying, we get the solution of (6.20)

$$\begin{aligned} u(x,t) &= A_x^{-1}K_t^{-1}\left[\frac{r}{(1-q^2r)}\frac{6(1-q^2r)}{q^5} + \frac{r}{(1-q^2r)}\frac{(1-q^2r)}{q(q^2+1)(1+r)}\right] \\ &= A_x^{-1}K_t^{-1}\left[\frac{6r}{q^5} + \frac{r}{q(q^2+1)(1+r)}\right] \\ &= x^3 + e^{-t}\sin x. \end{aligned}$$

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