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On Projective Motion in an N- Trirecurrent Finsler Space

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حول الحركة الإسقاطية في فضاء
N- Trirecurrent Finsler

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الملخص

في هذه الورقة البحثية تم الحصول على الشروط اللازمة والكافية للحركة الاسقاطية بأن تكون حركة أفينية، تم دراسة الحركة الإسقاطية في فضاء فنسلر-N والتي تحقق الخاصية الثلاثية المعاودة.

الكلمات المفتاحية: فضاء فنسلر ثلاثي المعاودة، الحركة الأفينية والحركة الإسقاطية.

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Abstract

In the present paper, the necessary and sufficient conditions for the projective motion to be affine motion are obtained. Projective motion is studied in trirecurrent Finsler space.

Keywords: Trirecurrent Finsler space, affine motion and projective motion.

1. Introduction

Let us consider an n-dimensional affine connected Finsler space F_n with a positively homogeneous metric function $F(x,y)$ of degree one in y^i.

The fundamental metric tensor g_{ij} of F_n is given by

(1.1) $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y)$.

The tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in i and j.

The covariant derivative of any vector field X^i with respect to x^j is given by [14]

(1.2) $\mathcal{B}_j X^i = \partial_j X^i - (\dot{\partial}_k X^i)\Pi^k_{rj} y^r + X^k \Pi^i_{kj}$,

where

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{jkr}.$$

The normal projective connection Π^i_{jk} and the connection parameters G^i_{jk} are positively homogeneous of degree zero in $y^i$and skew –symmetric [14].

The normal projective curvature tensor N^i_{jkh} is given by

(1.3) $N^i_{jkh} = 2\{\dot{\partial}_j \Pi^i_{[kh]} + \Pi^i_{rj[h}\Pi^r_{k]s} y^s + \Pi^i_{r[h}\Pi^r_{k]j}\}$,

where $[kh]$ represents skew – symmetric part. The derivatives $\dot{\partial}_j \Pi^i_{kh}$ is denoted by Π^i_{jkh}, and given by

$$\Pi^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + y^i G^r_{jkhr}),$$

where is symmetric in k and h only, and is positively homogeneous of degree -1 in y^i. That's tensor satisfies the following identity

(1.4) $\Pi^i_{jkh} y^j = 0$.

The normal projective curvature tensor N^i_{jkh} is skew-symmetric in its last two indices, i.e.

(1.5) $N^i_{jkh} = -N^i_{jhk}$.

Also, this tensor satisfies the following identity [13]

(1.6) $N^i_{jkh} + N^i_{khj} + N^i_{hjk} = 0$.

The normal projective curvature tensor N^i_{jkh} is related with Berwald curvature tensor H^i_{jkh} by

(1.7) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1} y^i \dot{\partial}_j H^r_{rkh}$,

where the curvature tensor H^i_{jkh} is positively homogeneous of degree zero in y^i and skew-symmetric in its last two indices which is given by

(1.8) $H^i_{jkh} \coloneqq 2\{\partial_{[h} G^i_{k]j} + G^r_{j[k} G^i_{h]r} - G^i_{rj[k} G^r_{h]}\}$.

The curvature tensor H^i_{jkh} satisfies the following identities

(1.9) a) $H^i_{jkh} y^j = H^i_{kh} = N^i_{jkh} y^j$ b) $H^i_{kh} y^k = H^i_h$

c) $H^i_i = (n-1)H$ d) $H^i_{ki} = H_k$

The commutation formula involving the above curvature tensor, and given by

(1.10) $2\,\mathcal{B}_{[l}\mathcal{B}_{m]} T^i_j = T^r_j N^i_{lmr} - T^i_r N^r_{lmj} - (\dot{\partial}_r T^i_j) N^r_{lms} y^s$.

In particular, the Berwald covariant derivative of the vector y^i vanishes identically, i.e.

(1.11) $\mathcal{B}_k y^i = 0$.

Definition 1.1.

The normal projective curvature tensor N^i_{jkh} satisfies the relation [8]

(1.12) $\mathcal{B}_l N^i_{jkh} = \lambda_l N^i_{jkh}, N^i_{jkh} \neq 0,$

where λ_l *is a non – zero recurrence vector field*, the space is called *recurrent Finsler space* ([4], [12], [13]).

Transvecting (1.12) by y^j, using (1.9) and (1.11), we get

(1.13) $\mathcal{B}_l H^i_{kh} = \lambda_l H^i_{kh}.$

Definition 1. 2.

The normal projective curvature tensor N^i_{jkh} satisfies the relation [8]

(1.14) $\mathcal{B}_m \mathcal{B}_l N^i_{jkh} = a_{lm} N^i_{jkh}, \quad N^i_{jkh} \neq 0,$

where a_{lm} *is a non – zero recurrence tensor field*, the space is called *birecurrent Finsler space* [1].

Transvecting (1.14) by y^j, using (1.9) and (1.11), we get

(1.15) $\mathcal{B}_m \mathcal{B}_l H^i_{kh} = a_{lm} H^i_{kh}.$

Definition 1. 3.

The normal projective curvature tensor N^i_{jkh} satisfies the relation

(1.16) $\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l N^i_{jkh} = c_{lmn} N^i_{jkh}, \ N^i_{jkh} \neq 0,$

where c_{lmn} *is a non – zero recurrence tensor field of order three*, the space is called *trirecurrent Finsler space* [1].

Let us consider an infinitesimal transformation

(1.17) $\bar{x}^i = x^i + \epsilon v^i(x^j),$

where ϵ is *an infinitesimal constant* and $v^i(x^j)$ is called *contravariant vector field* independent of y^i. Also, this transformation gives rise to a process of differentiation called *Lie- differentiation*.

Let X^i be an arbitrary contravariant vector field. Its Lie- derivative with respect to the above infinitesimal transformation is given by ([7], [9], [14])

(1.18) $L_v X^i = v^r \mathcal{B}_r X^i - X^r \mathcal{B}_r v^i + (\dot{\partial}_r X^i)\mathcal{B}_s v^r y^s$,

where the symbol L_v stands for the Lie-differentiation.

In view of (1.17) the Lie-derivatives of y^i and v^i with respect to above infinitesimal transformation vanish, i.e.

(1.19) $a)\ L_v y^i = 0$,

and

$b)\ L_v v^i = 0$.

Lie-derivative of an arbitrary tensor T^i_j with respect to the above infinitesimal transformation is given by

(1.20) $L_v T^i_j = v^r \mathcal{B}_r T^i_j - T^r_j \mathcal{B}_r v^i + T^i_r \mathcal{B}_j v^r + (\dot{\partial}_r T^i_j)\mathcal{B}_s v^r y^s$.

Lie-derivative of the normal projective connection parameters Π^i_{jk} is given by [14]

(1.21) $L_v \Pi^i_{jk} = \mathcal{B}_j \mathcal{B}_k v^i + N^i_{rjk} v^r + \Pi^i_{rjk} y^s \mathcal{B}_s v^r$.

The commutation formula for the operators $\mathcal{B}_k$, $\dot{\partial}_j$ and L_v are given by ([5],[12])

(1.22) $(L_v \mathcal{B}_k - \mathcal{B}_k L_v) X^i = X^h L_v \Pi^i_{kh} - (\dot{\partial}_r X^r) L_v \Pi^i_{kh} y^h$,

where X^i is *a contravariant vector field*.

The infinitesimal transformation (1.17) defines a motion, affine motion or projective motion if it preserves the distance between two points, parallelism of pair of vector or the geodesics, respectively. Necessary and sufficient conditions for the transformation (1.17) to be a motion, affine motion and projective motion are respectively given by [4]

(1.23) $L_v g_{ij} = 0,$

(1.24) $L_v \Pi^i_{kh} = 0,$

and

(1.25) $L_v \Pi^i_{jk} = \delta^i_j P_k + \delta^i_k P_j,$

where P_j is defined as

(1.26) $P_j = \dot{\partial}_j P,$

P being a scalar, positively homogeneous of degree one in y^i.

Transvecting (1.26) by y^j, using homogeneity of P_j [10], we get

(1.27) $P_j y^j = P.$

Let an infinitesimal transformation (1.17) be generated by a vector field $v^i(x^j)$.

The vector field $v^i(x^j)$ is called *contra*, *concurrent* and *special concircular* according as it satisfies

(1.28) $a)\ \mathcal{B}_k v^i = 0,$

$b)\ \mathcal{B}_k v^i = c\delta^i_k,$ c being a constant

and

$c)\ \mathcal{B}_k v^i = \rho\delta^i_k,$ ρ is not a constant, respectively. The affine motion generated by the above vectors is called *contra affine motion*, *concurrent affine motion* and *special concircular*, respectively.

Several results by authors extended to Finsler spaces of recurrent curvature by R. B. Misra, N. Kishore, and P. N. Pandey [6], A. Kumar, H. S. Shulka and R. P. Tripathi [2], S. P. Singh ([10], [11]) and others. C. K. Misra and D. D. S. Yadav [3] and S. P. Singh [12] discussed the affine motion in birecurrent non - Riemannian space.

2. Projective Motion in an N- Trirecurrent Finsler Space

Definition 2. 1.

A trirecurrent Finsler space characterized by (1.16) in which infinitesimal transformation (1.17) defines a projective motion is called *projective trirecurrent Finsler space* briefly denoted by $NT-P\bar{F}_n$.

Lie-derivative of the normal projective curvature tensor N^i_{jkh} satisfies the relation

(2.1) $L_v N^i_{jkh} = 2\mathcal{B}_{[j}P_{k]}\delta^i_h - 2\delta^i_{[j}\mathcal{B}_{k]}P_h - 2P\Pi^i_{[jk]h}$.

Transvecting (2.1) by y^j, using (1.4), (1.9a), (1.11), (1.19a) and (1.27), we get

(2.2) $L_v H^i_{kh} = \delta^i_h \mathcal{B}_k P - y^i \mathcal{B}_k P_h$.

In view of (1.5) and (1.11), applying Lie –derivative to (1.16) and observing (2.1), we get

(2.3) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l N^i_{jkh} = (L_v c_{lmn}) N^i_{jkh} + c_{lmn}\left(2\mathcal{B}_{[j}P_{k]}\delta^i_h - 2\delta^i_{[j}\mathcal{B}_{k]}P_h - 2P\Pi^i_{h[jk]}\right)$

Transvecting (2.3) by y^j, using (1.4), (1.9a), (1.11), (1.19a) and (1.27), we get

(2.4) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l H^i_{kh} = (L_v c_{lmn}) H^i_{kh} + c_{lmn}\left(\delta^i_h \mathcal{B}_k P - y^i \mathcal{B}_k P_h\right)$.

Thus, we conclude

Theorem 2.1.

In an $NT - P\bar{F}_n$, which admits a projective motion, the relations (2.3) and (2.4) holds.

Transvecting (2.3) by y^h, using (1.9b), (1.19a) and (1.27), we get

(2.5) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l H_k^i = (L_v c_{lmn}) H_k^i$.

Contracting the indices *i* and *k* in (2.5) and using (1.9c), we get

(2.6) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l H = (L_v c_{lmn}) H$.

Contracting the indices i and h in (2.4), using (1.9d), (1.11) and (1.27), we get

(2.7) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l H_k = (L_v c_{lmn}) H_k$.

Thus, we conclude

Theorem 2.2.

In an $NT - P\bar{F}_n$, which admits a projective motion, the division tensor of Berwald curvature tensor H_k^i, the scalar tensor H and the vector tensor H_k satisfied the relations (2.5), (2.6) and (2.7) respectively.

When the projective motion becomes an affine motion, the condition $L_v \Pi_{kh}^i = 0$ is satisfied. If we apply this condition in (1.25), we get

(2.8) $\delta_j^i P_k + \delta_k^i P_j = 0$.

Contracting the indices *i* and *k* in (2.8), we get

(2.9) $(1 + n) P_j = 0$,

which implies

(2.10) $P_j = 0$.

Conversely, if (2.10) is true, the equation (1.25) reduces to $L_v \Pi^i_{kh} = 0$. i.e. the necessary and sufficient condition for the infinitesimal transformation (1.17) which defines a projective motion to be an affine motion.

Using the equations (1.27) and (2.10) in (2.1) and (2.2), we get

(2.11) $L_v N^i_{jkh} = 0,$

and

(2.12) $L_v H^i_{kh} = 0.$

Again, Using the equations (1.27) and (2.10) in (2.3) and (2.4), we get

(2.13) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l N^i_{jkh} = (L_v c_{lmn}) N^i_{jkh}, \quad N^i_{jkh} \neq 0,$

and

(2.14) $L_v \mathcal{B}_n \mathcal{B}_m \mathcal{B}_l H^i_{kh} = (L_v c_{lmn}) H^i_{kh}.$

Thus, we conclude

Theorem 2.3.

In an $NT - P\bar{F}_n$, if the projective motion becomes an affine motion, then the relations (2.13) and (2.14) are necessarily true.

From (2.13), we get

(2.15) $(L_v c_{lmn}) N^i_{jkh} = 0, \;\; N^i_{jkh} \neq 0$, then we get

(2.16) $L_v c_{lmn} = 0$

Thus, we conclude

Theorem 2.4.

In an $NT - P\bar{F}_n$, if the projective motion becomes an affine motion, then the recurrence tensor field c_{lmn} satisfies the identity (2.16).

Applying skew – symmetric of (2.16), we get

(2.17) $L_v c_{lmn} + L_v c_{mnl} + L_v c_{nlm} = 0$

Thus, we conclude

Theorem 2.5.

In an $NT - P\bar{F}_n$, if the projective motion becomes an affine motion, then the recurrence tensor field c_{lmn} satisfies the identity (2.17).

3. Special cases

Let us consider an infinitesimal transformation generated by contra vector $v^i(x^j)$ characterized by (1.28a).

Taking the covariant derivative for (1.28a) with respect to x^j in sense of Berwald, we get

(3.1) $\mathcal{B}_j \mathcal{B}_k v^i = 0.$

Using equations (1.25), (1.28a), (1.4) and (3.1) in equation (1.21), we get

(3.2) $N^i_{jkh} v^h = \delta^i_j P_k + \delta^i_k P_j.$

Differentiating (3.2) covariant with respect to x^l, x^m and x^n in sense of Berwald and using (1.28a), we get

(3.3) $\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l N^i_{jkh} v^h = \delta^i_j \mathcal{B}_m \mathcal{B}_l P_k + \delta^i_k \mathcal{B}_m \mathcal{B}_l P_j.$

Using equations (1.16) and (3.2) in equation (3.3), we get

(3.4) $\delta^i_j(\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_k - c_{lmn} P_k) + \delta^i_k\left(\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_j - c_{lmn} P_j\right) = 0.$

Contracting the indies *i* and *j* in (3.4), we get

(3.5) $\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_k = c_{lmn} P_k.$

Thus, we conclude

Theorem 3.1.

In an $NT - P\bar{F}_n$, which admits projective motion, if the vector field $v^i(x^j)$ spans contra affine motion, then the vector P_k is trirecurrent.

Transvecting (3.5) by y^k, using (1.11) and (1.27), we get

(3.6) $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P = c_{lmn}P$.

Thus, we conclude

Theorem 3.2.

In an $NT - P\bar{F}_n$, which admits projective motion, if the vector field $v^i(x^j)$ spans contra affine motion, then the scalar function P is trirecurrent.

If we adopt the similar process for (1.28b), we get the following theorems:

Theorem 3.3.

In an $NT - P\bar{F}_n$, which admits projective motion, if the vector field $v^i(x^j)$ spans concircular affine motion, then the vector P_k is trirecurrent.

Theorem 3.4.

In an $NT - P\bar{F}_n$, which admits projective motion, if the vector *field $v^i(x^j)$ spans concircular affine motion, then the scalar function P is tirecurrent.*

Let us consider an infinitesimal transformation generated by contra vector $v^i(x^j)$ characterized by (1.28c).

Taking the covariant derivative for (1.28c) with respect to x^j in the sense of Berwald, we get

(3.7) $\mathcal{B}_j\mathcal{B}_k v^i = \mathcal{B}_j\rho\delta^i_k$.

Using equations (1.25), (1.28c), (1.4) and (3.7) in equation (1.21), we get

(3.8) $N^i_{jkh}v^h = \delta^i_j P_k + \delta^i_k P_j - \mathcal{B}_j\rho\delta^i_k$.

Differentiating (3.8) covariant with respect to x^l, x^m and x^n, respectively in the sense of Berwald and using (1.28c), we get

(3.9) $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l N^i_{jkh} v^h = \delta^i_j \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_k + \delta^i_k \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_j - \delta^i_k \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\mathcal{B}_j \rho.$

Using equations (1.16) and (3.8) in equation (3.9), we get

(3.10) $\delta^i_j(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_k - c_{lmn}P_k) + \delta^i_k(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_j - c_{lmn}P_j) -$

$\delta^i_k(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\mathcal{B}_j\rho - c_{lmn}\mathcal{B}_j\rho) = 0.$

Contracting the indies *i* and *j* in (3.10), we get

(3.11) $(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_k - c_{lmn}P_k) - (\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\mathcal{B}_k\rho - c_{lmn}\mathcal{B}_k\rho) = 0.$

If

(3.12) $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\mathcal{B}_k\rho - c_{lmn}\mathcal{B}_k\rho = 0.$

Then we can written equation (3.11) as:

(3.13) $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P_k = c_{lmn}P_k.$

Thus, we conclude

Theorem 3.5.

In an $NT - P\bar{F}_n$, which admits projective motion, the vector field $v^i(x^j)$ spans special concircular affine motion then the vector P_k satisfied the trirecurrent property if condition (3.12) hold.

Transvecting (3.11) by y^k, using (1.11) and (1.27), we get

(3.14) $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P = c_{lmn}P.$

Thus, we conclude

Theorem 3.6.

In an $NT - P\bar{F}_n$, which admits projective motion, if the vector field $v^i(x^j)$spans special concircular affine motion, then the scalar function P is trirecurrent.

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